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Clifford Fields and the Relativistic Equation of the Nucleon

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A relativistic equation for free fields which take their values in the Clifford algebra associated with the Minkowski metric is shown to be interpretable as the equation of the nucleon. The internal symmetry group $SU(2)$ arises naturally from the associative algebra structure of the representation space. The latter structure can be used to construct coupling terms consistent with the transformation properties of the interacting fields; in particular, it allows the familiar couplings of the nucleon field with the electromagnetic field and with the π-meson field.

1. INTRODUCTION

Let $\mathscr{C}$ denote the complex Clifford algebra associated with the Minkowski metric, i.e., the abstract 16-dimensional associative complex algebra with unit with four generators e^1, e^2, e^3, e^4 and relations

$$e^i \vee e^h + e^h \vee e^i = 2\eta^{ih} \qquad (i, h = 1, 2, 3, 4) \tag{1}$$

where $\vee$ denotes the product, and the numerical matrix $(\eta^{ih}) \equiv \mathrm{diag}(-1, -1, -1, 1)$ represents the components of the metric in canonical form.

In a previous paper (Cantoni, 1997), two distinct representations of the Lie algebra $so(3,1)$ carried by $\mathscr{C}$ were considered, and it was remarked that both satisfy the conditions which guarantee the invariance of a field equation of the form

$$e^h \vee \frac{\partial \phi}{\partial x^h} + k\phi = 0 \tag{2}$$

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with respect to the proper Lorentz group $\mathscr{L}_0$. In equation (2) ϕ is a $\mathscr{C}$-valued function of the space-time Cartesian coordinates (x^1, x^2, x^3, x^4) and k is a constant.

Thus equation (2) admits two distinct interpretations as a relativistic equation, depending on the choice of the representation.

If the field ϕ is transformed according to the representation of $\mathscr{L}_0$ generated by the representation of the Lie algebra $so(3,1)$, which was denoted by "ad" (*adjoint*) in Cantoni (1997), it can be interpreted as an inhomogeneous differential form and will be called a *Kähler field*. In this case, as a relativistic equation, equation (2) is identical with the Kahler equation (Talebaoui, 1995), which represents bosons and decomposes into four distinct equations of Duffin–Kemmer type (Cantoni, 1996).

If, on the other hand, the field ϕ is transformed according to the two-valued representation of $\mathscr{L}_0$ generated by the representation of $so(3,1)$, which was denoted by "reg" (*regular*), it will be called a *Clifford field*, and equation (2), as a relativistic equation, represents fermions and decomposes into four equations equivalent to the Dirac equation (Talebaoui, 1995).

In this paper we shall be exclusively concerned with the latter interpretation of equation (2). Thus the fields ϕ will be Clifford fields, assumed to transform according to the two-valued representation of $\mathscr{L}_0$ just mentioned, extended to the entire homogeneous Lorentz group $\mathscr{L}$ by taking the operators of left multiplication by $\pm e^4$ and by $\pm e^1 \vee e^2 \vee e^3$ as representatives of the space reflection and time reflection, respectively. As in Cantoni (1997), the equation itself will be referred to as the *Clifford equation*.

We shall show that, by means of the Clifford algebra structure of the representation space $\mathscr{C}$, the decomposition of the Clifford equation with respect to the homogeneous Lorentz group can be made in two steps, the first of which is natural, while the second is not. The first step splits the 16-component Clifford equation into two mutually equivalent eight-component *reduced Clifford equations*. The second step (further reduction of the reduced Clifford equations into pairs of Dirac equations) is carried out subordinately to the choice of an additional structure.

Each reduced Clifford equation has symmetries allowing its interpretation as the relativistic equation of the free nucleon. While the symmetry with respect to the Poincaré group was imposed by construction, the internal symmetry group $SU(2)$ arises naturally from the associative algebra structure of the representation space. The latter structure is used to construct the simplest coupling of a Clifford field to an unspecified Kahler field, and to express the familiar interaction Lagrangians of the nucleon field with the electromagnetic field and with the π-meson field.

2. THE REGULAR REPRESENTATION OF THE HOMOGENEOUS LORENTZ GROUP AND ITS NATURAL REDUCTION

Setting $e^{ih} \equiv e^i \vee e^h$ (for i, h ranging from 1 to 4 and $i \neq h$), with the help of the relations $e^{\alpha\beta} \vee e^{\alpha\beta} = -1$ and $e^{\alpha 4} \vee e^{\alpha 4} = 1$ (for α and β ranging from 1 to 3), the representation "reg" of the real Lie algebra $\mathfrak{so}(3,1)$ on the (real or complexified) Clifford algebra $\mathscr{C}$, described in Cantoni (1997), can be readily exponentiated. One gets the two-valued representation of the homogeneous proper Lorentz group $\mathscr{L}_0$ in which the one-parameter subgroups of space rotations and boosts generated by the elements $\frac{1}{2}\, e^{\alpha\beta}$ and $\frac{1}{2}\, e^{\alpha 4}$, respectively, are represented by the one-parameter subgroups of operators of left-multiplication

$$\left(\cos\frac{\vartheta}{2} + e^{\alpha\beta}\sin\frac{\vartheta}{2}\right)\vee \tag{3}$$

and

$$\left(\cosh\frac{\xi}{2} + e^{\alpha 4}\sinh\frac{\xi}{2}\right)\vee \tag{4}$$

Setting $p \equiv e^4$ and $t \equiv e^1 \vee e^2 \vee e^3$, the operators $\mp\, p\vee$ and $\mp\, t\vee$ extend the representation of $\mathscr{L}_0$ to a two-valued representation of the entire homogeneous Lorentz group $\mathscr{L}$, with $\mp\, p\vee$ and $\mp\, t\vee$ corresponding to the space reflection and time reflection, respectively. With this extension of the representation, the Clifford equation satisfies all the conditions of invariance with respect to $\mathscr{L}$ (Naimark, 1962, Ch. 4, §3, n. 1).

We now consider the complex Clifford algebra $\mathscr{C}$. The linear combinations of products of the generators with real coefficients will be called *real elements* of $\mathscr{C}$.

Since all the representatives of $\mathscr{L}_0$ are *even* real elements (i.e., they belong to the subspace $\mathscr{C}_0$ generated by all the elements of degree 0, 2, or 4), the *even* subspace $\mathscr{C}_0$ and the *odd* subspace $\mathscr{C}_1$ (generated by all the elements of degree 1 or 3) are invariant under $\mathscr{L}_0$.

Similarly, since the operator $e^5\vee$ (with $e^5 \equiv e^1 \vee e^2 \vee e^3 \vee e^4$) has eigenvalues $\mp\, \mathrm{i}$ and commutes with the even elements of $\mathscr{C}$, the subspaces $\mathscr{C}^+$ and $\mathscr{C}^-$ of $\mathscr{C}$ constituted of eigenvectors of $e^5 \vee$ belonging to the eigenvalues i and $-\mathrm{i}$, respectively, are invariant under $\mathscr{L}_0$.

Hence the four subspaces $\mathscr{C}_0^+ \equiv \mathscr{C}_0 \cap \mathscr{C}^+$, $\mathscr{C}_0^- \equiv \mathscr{C}_0 \cap \mathscr{C}^-$, $\mathscr{C}_1^+ \equiv \mathscr{C}_1 \cap \mathscr{C}^+$, and $\mathscr{C}_1^- \equiv \mathscr{C}_1 \cap \mathscr{C}^-$ are invariant under $\mathscr{L}_0$. On the other hand, since p and t are odd and anticommute with e^5, the subspaces $\Gamma \equiv \mathscr{C}_0^+ + \mathscr{C}_1^-$ and $\Gamma' \equiv \mathscr{C}_0^- + \mathscr{C}_1^+$ of $\mathscr{C}$ are invariant under $\mathscr{L}$.

The subspaces Γ and Γ' are also invariant with respect to the four operators $e^h \vee$ of left multiplication by e^h, so that they are invariant with respect to the differential operator of equation (2). Therefore the Clifford equation decomposes into two equations of the same form, whose fields take their values in Γ and in Γ', respectively. As we shall see, the two *reduced Clifford equations* so obtained turn out to be equivalent.

3. FURTHER REDUCTION OF THE CLIFFORD EQUATION

The above decomposition of the representation of $\mathcal{L}$ and of the original equation is *natural* with respect to $\mathcal{C}$ in the sense that it is entirely determined by the Clifford algebra structure of the representation space.

We shall now study a distinguished class of further decompositions of the reduced equations. The decompositions of this class, which will be called *canonical*, are adapted to the Clifford algebra structure of the representation space in the sense that their invariant subspaces are orthogonal minimal left ideals of $\mathcal{C}$. The canonical decompositions are in one-to-one correspondence with a special class of elements of $\mathcal{C}$, which constitute a homogeneous space of $Sl(2, C)$ under a natural action of this group. The restriction of this action to $SU(2)$ is related to the "internal" symmetry group of the reduced Clifford equations.

We start by noting that the natural decomposition of $\mathcal{C}$ considered above can be conveniently described in terms of the two elements $p_5 \equiv \frac{1}{2}(1 - \mathtt{i}e^5)$ and $q_5 \equiv \frac{1}{2}(1 + \mathtt{i}e^5)$, which are complementary orthogonal idempotents of $\mathcal{C}$ (i.e., $p_5 + q_5 = 1$, $p_5 \vee p_5 = p_5$, $q_5 \vee q_5 = q_5$, $p_5 \vee q_5 = q_5 \vee p_5 = 0$). It is easily checked that the operators $\vee p_5$ and $\vee q_5$ of right multiplication by these elements are projection operators on the subspaces Γ and Γ', respectively.

In order to obtain a further decomposition of Γ (or Γ') into invariant subspaces, which turn out to be minimal left ideals of $\mathcal{C}$, we choose any real element $\mathtt{e}$ of $\mathcal{C}$ of degree 2 and such that $\mathtt{e} \vee \mathtt{e} = -1$. Setting

$$p_{\mathtt{e}} \equiv \tfrac{1}{2}(1 - \mathtt{ie}), \qquad q_{\mathtt{e}} \equiv \tfrac{1}{2}(1 + \mathtt{ie})$$

we get the following set of four complementary orthogonal primitive idempotents:

$$\begin{aligned}
\mathtt{a}_{\mathtt{e}} &\equiv p_5 \vee q_{\mathtt{e}} = \tfrac{1}{4}(1 - \mathtt{i}e^5)(1 + \mathtt{ie}) \\
\mathtt{b}_{\mathtt{e}} &\equiv p_5 \vee p_{\mathtt{e}} = \tfrac{1}{4}(1 - \mathtt{i}e^5)(1 - \mathtt{ie}) \\
\mathtt{a}'_{\mathtt{e}} &\equiv q_5 \vee p_{\mathtt{e}} = \tfrac{1}{4}(1 + \mathtt{i}e^5)(1 - \mathtt{ie}) \\
\mathtt{b}'_{\mathtt{e}} &\equiv q_5 \vee q_{\mathtt{e}} = \tfrac{1}{4}(1 + \mathtt{i}e^5)(1 + \mathtt{ie})
\end{aligned} \tag{5}$$

The minimal left ideals determined by the projection operators $\vee\, \mathtt{a}_{\mathtt{e}}$ and $\vee\, \mathtt{b}_{\mathtt{e}}$ span Γ. In a similar way, $\vee \mathtt{a}'_{\mathtt{e}}$ and $\vee \mathtt{b}'_{\mathtt{e}}$ are related to Γ'.

Consider now the special case $\mathtt{e} = e^{12}$. The corresponding elements (5) will be written $\mathtt{a}_{12}$, $\mathtt{b}_{12}$, $\mathtt{a}'_{12}$, $\mathtt{b}'_{12}$. Since the element $\mathtt{a}_{12}$ is an eigenvector of the operator $H_3 \equiv \frac{1}{2}\, \mathtt{i} e^{12} \vee$ with eigenvalue 1/2 and of the operator $F_3 \equiv \frac{1}{2}\, \mathtt{i} e^{34} \vee$ with eigenvalue $-\mathtt{i}/2$, under the action of $\mathscr{L}_0$ it generates a two-dimensional invariant subspace of $\mathscr{C}_0^+$ carrying a representation of $\mathscr{L}_0$ of type $(k_0, \mathrm{c}) = (1/2, -3/2)$ in Naimark's classification (Naimark, 1962, Ch. 3, §2, n. 3, p. 98), and under the action of $\mathscr{L}$ it generates a four-dimensional invariant subspace of Γ carrying a representation equivalent to the representation of $\mathscr{L}$ on Dirac spinors. This invariant subspace coincides with the minimal left ideal associated with $\mathtt{a}_{12}$.

Similarly, since $\mathtt{b}_{12}$ is an eigenvector of H_3 with eigenvalue $-1/2$ and of F_3 with eigenvalue $\mathtt{i}/2$, under the action of $\mathscr{L}_0$ it generates an invariant subspace of $\mathscr{C}_0^+$ carrying a representation of $\mathscr{L}_0$ of the same type as above, and again, under the action of $\mathscr{L}$, it generates an invariant subspace of Γ carrying a representation of $\mathscr{L}$ equivalent to the Dirac representation. This invariant subspace coincides with the minimal left ideal associated with $\mathtt{b}_{12}$.

The two invariant subspaces of Γ arising from this decomposition are invariant under the differential operator of equation (2), so that the reduced Clifford equation on Γ decomposes into two equations, equivalent to the Dirac equation. Replacing $\mathtt{a}_{12}$ and $\mathtt{b}_{12}$ by $\mathtt{a}'_{12}$ and $\mathtt{b}'_{12}$, one gets a similar decomposition of Γ'.

The mutual equivalence of the two reduced Clifford equations is exhibited by the operator of right-multiplication

$$\vee\, e^4\colon \quad \phi \rightarrow \phi \vee e^4$$

which commutes with the representatives of $Sl(2, C)$, is its own inverse, and maps the Γ-valued solutions of (2) onto the Γ'-valued solutions. Thus from now on we shall only need to be concerned with one of the reduced equations, say with the equation on Γ.

We now show that there is a one-to-one correspondence between the canonical decompositions (5) and the elements $\mathtt{e}$ of $\mathscr{C}$ with the properties of being real, of degree 2, and such that $\mathtt{e} \vee \mathtt{e} = -1$. In fact, given the decomposition (5), let $\mathtt{f}$ be an element with the same properties and giving rise to the same decomposition, so that $\mathtt{a}_{\mathtt{e}} = \mathtt{a}_{\mathtt{f}}$. From this equation one gets the relation $(1 - \mathtt{i} e^5)(\mathtt{e} - \mathtt{f}) = 0$, so that $\mathtt{e} - \mathtt{f}$ must belong to Γ', and therefore, being even, to $\mathscr{C}_0^-$. But $\mathscr{C}_0^-$ is constituted of eigenvectors of $e^5 \vee$, and $e^5 \vee$ has no real eigenvector (since it transforms real vectors into real vectors while its eigenvalues are pure imaginary); therefore $\mathtt{e} - \mathtt{f} = 0$ and $\mathtt{f}$ coincides with $\mathtt{e}$. Thus the set $\mathscr{D}$ of canonical decompositions is identified with the set of all the real elements $\mathtt{e}$ of $\mathscr{C}$ of degree 2 and such that $\mathtt{e} \vee \mathtt{e} = -1$.

4. THE ACTION OF $Sl(2, C)$ ON THE SET OF CANONICAL DECOMPOSITIONS

Let σ be a representative of an element of $Sl(2, C)$ in the regular representation. If e is an element of $\mathcal{D}$, the element

$$\mathrm{e}' = \sigma \vee \mathrm{e} \vee \sigma^{-1} \tag{6}$$

also belongs to $\mathcal{D}$, since σ is a product of elements of the form (3) or (4). Therefore $\mathcal{D}$ is invariant under the adjoint action of $Sl(2, C)$ on $\mathcal{C}$, which is given by

$$\mathrm{a} \to \sigma \vee \mathrm{a} \vee \sigma^{-1} \qquad (\mathrm{a}\ \varepsilon\ \mathcal{C}) \tag{7}$$

We now prove that $\mathcal{D}$ is a single orbit under this action by showing that any element $\mathrm{e} \in \mathcal{D}$ can be transformed into the particular element e^{12} considered in the previous section.

To see this, let us recall that under the adjoint action of $\mathcal{L}$, the Clifford algebra $\mathcal{C}$ can be identified with the exterior algebra of Minkowski space, with the elements of degree 2 corresponding to the exterior 2-forms. Since an exterior 2-form has the transformation properties of an electromagnetic tensor, for the element $\mathrm{e} = \alpha_{ih}e^{ih}$ we can expressively introduce electromagnetic notations by setting

$$(\alpha_{ih}) \equiv \begin{pmatrix} 0 & H_3 & -H_2 & E_1 \\ -H_3 & 0 & H_1 & E_2 \\ H_2 & -H_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix}$$

with these notations the assumption $\mathrm{e} \vee \mathrm{e} = -1$ entails the two relations

$$E^2 - H^2 = -1 \tag{8}$$

and

$$\mathbf{E}\cdot\mathbf{H} = 0 \tag{9}$$

where E^2 and H^2 denote the squared moduli of the "electric vector" and of the "magnetic vector," and $\mathbf{E}\cdot\mathbf{H}$ denotes their Euclidian scalar product. These relations are Lorentz-invariant, and due to (9) it is possible, with a spatial rotation, to go to a Lorentz frame where $H_1 = H_2 = 0$, $E_2 = E_3 = 0$, so that the representative matrix is reduced to $(0, H, 0, E)$ on the first row, $(0, -H, 0, -E)$ on the first column, and zero elsewhere. Subsequently, setting $E = \beta H$ [where $|\beta| < 1$ on account of (8)], with a boost with velocity βc along the second axis one can reach a Lorentz frame where E vanishes, while the new value of H is equal to 1 on account of (8), and the representative matrix of our 2-form is reduced to the matrix (α_{ih}) corresponding to the element e^{12}. Since the action of an element of $\mathcal{L}$ on the exterior forms identified with $\mathscr{C}$ is the same as the action of a pair of opposite elements of $Sl(2, C)$ in the adjoint representation, the given element e can indeed be transformed into e^{12} as stated.

If σ is a representative of an element of $Sl(2, C)$ such that $\sigma \vee \mathrm{e} \vee \sigma^{-1} = e^{12}$, the transformation $\alpha \to \sigma^{-1} \vee \alpha \vee \sigma$ applied to the 16 elements of the basis

$$\{1, e^1, e^2, \ldots, e^{12}, e^{13}, \ldots, e^{123}, e^{124}, \ldots, e^{1234}\}$$
$$(e^{12} \equiv e^1 \vee e^2, e^{123} \equiv e^1 \vee e^2 \vee e^3, \text{etc.}) \tag{10}$$

gives a new basis of the same kind, with e equal to the element $e'^{12} \equiv \sigma^{-1} \vee e^{12} \vee \sigma$ of the new basis. Thus, dropping the prime, we see that any canonical decomposition can be associated with the element e^{12} of some appropriate basis.

5. THE LAGRANGIAN AND ITS EXTERNAL SYMMETRIES

Consider the basis (10) of $\mathscr{C}$, and denote by $\mathscr{E}^h$ ($h = 1, 2, 3, 4$) the representative 16×16 matrix of the linear operators $e^h \vee$ on $\mathscr{C}$ in this basis. Since $e^4 \vee e^4 = 1$, $\mathscr{E}^4$ is nonsingular, and it is readily checked that it is real and symmetric, so that it can be used to define a scalar product in $\mathscr{C}$ by setting

$$(\gamma, \phi) \equiv \overline{\Gamma}^t \mathscr{E}^4 \Phi \qquad (\phi, \gamma \in \mathscr{C}) \tag{11}$$

where Φ and Γ are the column matrices of the components of ϕ and γ, respectively, with the notation $\overline{M}$ for the complex conjugate and M^t for the transpose of any matrix M. The scalar product (11) is linear with respect to ϕ, antilinear with respect to γ, and is not positive-definite.

The one-dimensional subgroups (3) and (4) are represented by the matrices

$$\cos \frac{\vartheta}{2} \cdot I + \sin \frac{\vartheta}{2} \cdot \mathscr{E}^\alpha \mathscr{E}^\beta$$

and

$$\cosh \frac{\xi}{2} \cdot I + \sinh \frac{\xi}{2} \cdot \mathscr{E}^\alpha \mathscr{E}^4$$

where I denotes the unit 16×16 matrix. Using the fact that the matrices

$\mathscr{E}^{\alpha}$ ($\alpha = 1, 2, 3$) turn out to be antisymmetric, and that the $\mathscr{E}^h$ ($h = 1, 2, 3, 4$) satisfy the same anticommutation relations as the operators $e^h \vee$ of $\mathscr{C}$ that they represent, one can check that the scalar product (11) is invariant with respect to the action of the operators (3) and (4), and therefore with respect to the regular action of $Sl(2, C)$ on $\mathscr{C}$. It is also invariant with respect to the action of the operator $e^4 \vee$, represented by the matrix $\mathscr{E}^4$.

Denoting now by Φ the column matrix of the components of a Clifford field or of a reduced Clifford field, the function

$$L = \frac{\mathrm{i}}{2}\{(\Phi, \mathscr{E}^h\partial_h\Phi) - (\mathscr{E}^h\partial_h\Phi, \Phi)\} + \mathrm{i}k(\Phi, \Phi)$$

or, in basis-independent form,

$$L = \frac{\mathrm{i}}{2}\{(\phi, e^h \vee \partial_h\phi) - (e^h \vee \partial_h\phi, \phi)\} + \mathrm{i}k(\phi, \phi) \tag{12}$$

is a Lagrangian for the Clifford equation and for the reduced Clifford equation. The translational invariance of the action functional and its invariance with respect to the regular action of $Sl(2, C)$ and $e^4 \vee$ on $\mathscr{C}$ [due to the invariance properties of the scalar product (11)] imply that the Clifford equation and the reduced Clifford equation are invariant with respect to the associated two-valued action of the orthochronous Poincaré group, as we already know from our original construction of the equation. This means that for any solution $\phi(\mathbf{x})$ of the equation and for any Poincaré transformation $\mathbf{x} \to \Lambda\mathbf{x} + \mathbf{a}$ (where $\mathbf{a}$ and Λ denote the translation vector and the homogeneous part of the transformation, respectively) the field transformation

$$\phi(\mathbf{x}) \to \phi'(\mathbf{x}) \equiv \sigma_\Lambda \vee \phi(\Lambda^{-1}(\mathbf{x} - \mathbf{a})) \tag{13}$$

gives rise to a new solution of the equation. In (13) $\sigma_\Lambda \vee$ denotes the representative of either element of $Sl(2, C)$ associated with Λ in the regular representation.

We shall now see that besides these *external* symmetries, related to space-time transformations, the Lagrangian $\mathscr{L}$ possesses an *internal* symmetry group isomorphic to $SU(2)$.

6. THE INTERNAL SYMMETRIES

The restriction of the regular action of $Sl(2, C)$ on $\mathscr{C}$ to the subgroup $SU(2)$ is generated by the one-parameter subgroups of linear transformations (3):

$$\left(\cos\frac{\vartheta}{2} + e^{\alpha\beta}\sin\frac{\vartheta}{2}\right)\vee \qquad (\alpha,\ \beta = 1,\ 2,\ 3;\ \alpha < \beta) \tag{14}$$

This external (left) action of the generic element $u \in Sl(2, C)$ on $\mathscr{C}$ has the form

$$a \to u \vee a \qquad (a \in \mathscr{C}) \tag{15}$$

where u is a product of elements of the form (14).

We now define the following *internal* (left) action of $SU(2)$ on Clifford fields:

$$u\colon \quad \phi(\mathbf{x}) \to \phi'(\mathbf{x}) \equiv \phi(\mathbf{x}) \vee u^{-1} \tag{16}$$

with the same meaning of u as above. [Notice that, in contrast with the transformation (13), this transformation does not affect the space-time coordinates. Notice also that, since $\vee\, u^{-1}$ is an operator of *right* multiplication in $\mathscr{C}$, it does not preserve the subspaces arising from the canonical decompositions of the fields, which are *left* ideals.]

If we denote by $\mathscr{A}$ the matrix representation of the generic operator of right multiplication $\vee\, a$ in $\mathscr{C}$, the action (16) is described by

$$\Phi \to \Phi' \equiv \mathcal{U}^{-1}\Phi \tag{17}$$

where $\mathcal{U}$ denotes the matrix of the transformation $\vee\, u$ and the dependence of Φ on $\mathbf{x}$ is understood. In particular, for the generating elements

$$u = \cos\frac{\vartheta}{2} + e^{\alpha\beta}\sin\frac{\vartheta}{2} \qquad (\alpha,\ \beta = 1,\ 2,\ 3;\ \alpha \neq \beta) \tag{18}$$

we have

$$\mathcal{U} = \cos\frac{\vartheta}{2}\cdot I + \sin\frac{\vartheta}{2}\cdot\mathscr{E}^{\beta}\mathscr{E}^{\alpha}$$

$$\mathcal{U}^{-1} = \cos\frac{\vartheta}{2}\cdot I - \sin\frac{\vartheta}{2}\cdot\mathscr{E}^{\beta}\mathscr{E}^{\alpha}$$

Since the matrices $\mathscr{E}^1, \mathscr{E}^2, \mathscr{E}^3$ (like the matrices $\mathscr{E}^1, \mathscr{E}^2, \mathscr{E}^3$) are skew-symmetric and anticommuting, the matrices $\mathscr{E}^{\beta}\mathscr{E}^{\alpha}$ are also skew-symmetric, so that one has $\mathcal{U}^t = \mathcal{U}^{-1}$ and the matrices $\mathcal{U}$, which are real, are orthogonal. Consequently, since the matrices $\mathcal{U}$ commute with $\mathscr{E}^4$, the scalar product (11) is invariant under the action (16) of $SU(2)$ on the fields:

$$\begin{aligned}(\gamma \vee u^{-1},\ \phi \vee u^{-1}) &= \overline{\Gamma}^t(\mathcal{U}^{-1})^t\,\mathscr{E}^4\mathcal{U}^{-1}\Phi \\ &= \overline{\Gamma}^t\mathcal{U}\mathscr{E}^4\mathcal{U}^{-1}\Phi = \overline{\Gamma}^t\mathscr{E}^4\mathcal{U}\mathcal{U}^{-1}\Phi = (\gamma,\ \phi)\end{aligned} \tag{19}$$

[Notice that, although the action (16) of $SU(2)$ on the fields can be extended to the whole group $Sl(2, C)$, the scalar product (11) is not invariant with respect to the action of the elements which do not belong to $SU(2)$. In order to get an $Sl(2, C)$-invariant scalar product under this action, one would have to replace $\mathscr{E}^4$ by $\mathscr{E}^4$ in the definition (11).]

The invariance of the scalar product under the internal action (16) entails that $SU(2)$ is a symmetry group of the Lagrangian (12). Since the elements $u \in \mathscr{C}$ associated with the elements $u \in SU(2)$ are even, and therefore commute with p_5 and q_5 (see Section 3), the decomposition of the Clifford fields into reduced Clifford fields via the projection operators $\vee\, p_5$ and $\vee\, q_5$ is preserved by the internal action, though the finer canonical decompositions arising from the subsequent application of projection operators of the form $\vee\, p_{\mathtt{e}}$ and $\vee\, q_{\mathtt{e}}$ are not.

If the reduced Clifford equation on Γ is interpreted as the relativistic equation of the nucleon, and by means of the decomposition determined by a selected element $\mathtt{e} \in \mathscr{D}$ the components $\phi_{\mathrm{P}} \equiv \phi \vee \mathtt{b}_{\mathtt{e}}$ and $\phi_{\mathrm{N}} \equiv \phi \vee \mathtt{a}_{\mathtt{e}}$ of the generic field Φ are interpreted as proton states and neutron states, respectively, the operator $\tau_3 \equiv \vee\,(-\mathtt{ie})$ acts as the isospin operator characterized by the property of multiplying by 1 the proton states and by -1 the neutron states. In fact one has

$$\begin{aligned}\tau_3\phi_{\mathrm{P}} = \tau_3(\phi \vee \mathtt{b}_{\mathtt{e}}) &= -\phi \vee \mathtt{b}_{\mathtt{e}} \vee \mathtt{ie}\\ &= -\phi \vee \tfrac{1}{4}(1 - \mathtt{ie}^5) \vee (1 - \mathtt{ie}) \vee \mathtt{ie}\\ &= -\tfrac{1}{4}\, \phi \vee (1 - \mathtt{ie}^5) \vee (\mathtt{ie} - 1) = \phi \vee \mathtt{b}_{\mathtt{e}} = \phi_{\mathrm{P}}\end{aligned}$$

and similarly

$$\tau_3\phi_{\mathrm{N}} = \tau_3(\phi \vee \mathtt{a}_{\mathtt{e}}) = -\phi \vee \mathtt{a}_{\mathtt{e}} = -\phi_{\mathrm{N}}$$

Selecting $\mathtt{e} = e^{12}$ and setting $\tau_1 = \vee\, \mathtt{i}e^{23}$ and $\tau_2 = \vee\, \mathtt{i}e^{31}$, we see that the operators τ_1, τ_2, and τ_3 have the characteristic properties of the isospin operators usually denoted by the same symbols τ_1, τ_2, τ_3 in the literature (see, e.g., Bogoliubov and Shirkov, 1959, §33.1; Roman, 1965, pp. 544–545; Greiner and Muller, 1993, p. 91). In fact, setting $\tau_+ = \frac{1}{2}(\tau_1 + \mathtt{i}\tau_2)$ and $\tau_- = \frac{1}{2}(\tau_1 - \mathtt{i}\tau_2)$, we have $\tau_+^2 = 0$, $\tau_-^2 = 0$, $\tau_3 = \tau_+\tau_- - \tau_-\tau_+$, $[\tau_3, \tau_+] = 2\tau_+$, and $[\tau_3, \tau_-] = -2\tau_-$. In particular, τ_+ transforms neutron fields into proton fields, while τ_- acts conversely.

Since right translations commute with left ones, the $SU(2)$ action is Lorentz-invariant, as isospin is.

7. INTERACTIONS

The interaction with an external electromagnetic field with four-potential $\bar{A}$ is described by equation (2) supplemented with an interaction term of the form $-qA_h\, e^h \vee \phi \vee \mathrm{b_e}$, where q denotes the electric charge multiplied by a factor depending on the system of units adopted:

$$e^h \vee \frac{\partial \phi}{\partial x^h} + \mathrm{i} q A_h\, e^h \vee \phi \vee \mathrm{b_e} + k\phi = 0 \tag{20}$$

Setting $k \equiv \mathrm{i}\mu$, where μ is a real constant interpreted as the rest mass of the nucleon multiplied by $2\pi c/h$ (c is the velocity of light, h is Planck's constant), upon multiplication by $\mathrm{b_e}$ on the right equation (20) reduces to the Dirac equation with minimal electromagnetic coupling on the proton subspace:

$$\mathrm{i}e^h \vee \left(\frac{\partial}{\partial x^h} + \mathrm{i} q A_h\right)\phi_{\mathrm{P}} - \mu\phi_{\mathrm{P}} = 0$$

while upon multiplication by $\mathrm{a_e}$ on the right, it reduces to equation (2) on the neutron subspace:

$$\mathrm{i}e^h \vee \frac{\partial \phi_{\mathrm{N}}}{\partial x^h} - \mu\phi_{\mathrm{N}} = 0$$

The interaction term of equation (20) can be derived from the Lagrangian

$$L_{\mathrm{int}} = -q(\phi \vee \mathrm{b_e},\, A_h e^h \vee \phi \vee \mathrm{b_e})$$

The presence of an external electromagnetic field introduces the preferred decomposition e which establishes the distinction between proton and neutron states.

If a hypothetical interaction of the Clifford field with an external Kahler field $\underset{\mathrm{K}}{\Psi}$ is assumed to be described by an additional term of the form $\mathrm{k}\underset{\mathrm{K}}{\Psi}\vee\phi$ (where k is a coupling constant), no preferred decomposition occurs and the resulting equation

$$e^h \vee \frac{\partial \phi}{\partial x^h} + \mathrm{k}\underset{\mathrm{K}}{\Psi} \vee \phi + \mathrm{i}\mu\phi = 0 \tag{21}$$

has the same external and internal symmetries as the equation of the free field. In fact, since the transformation law of a Kahler field under $Sl(2, C)$ is given by

$$\underset{\mathrm{K}}{\Psi}(x) \to \sigma_\Lambda \vee \underset{\mathrm{K}}{\Psi}(\Lambda^{-1}(\mathbf{x} - \mathbf{a})) \vee \sigma_\Lambda^{-1}$$

(with the notation of Section 5), while the field ϕ transforms according to (13), the interaction term $\mathrm{k}\underset{\mathrm{K}}{\Psi}\phi$ is indeed a Clifford field (provided that, under a Poincaré transformation, the transformed product of the fields is defined as the product of the transformed factors) and the interaction Lagrangian

$$L_{\rm int} = \mathrm{k}(\phi, \underset{K}{\Psi} \vee \phi)$$

has the appropriate invariance properties.

A different kind of coupling appears in the description of the interaction of the nucleon field with the π-meson field within the present framework. Besides the Clifford fields and the Kahler fields, consider fields of yet another kind, still $\mathscr{C}$-valued, but associated with the trivial action of $Sl(2, C)$ (i.e., the action in which every element of the group acts as the identity operator on $\mathscr{C}$). If $a_1, a_2, \ldots, a_n$ are elements of $\mathscr{C}$, and $f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})$ are scalar (or pseudoscalar) functions on Minkowski space-time, a typical field of this kind, which will be called a *$\mathscr{C}$-valued field of scalar* (or *pseudoscalar*) *type*, is given by the $\mathscr{C}$-valued function $\Sigma f_i(\mathbf{x})\, a_i$, and transforms under the Poincaré transformation $\mathbf{x} \to \Lambda\mathbf{x} + \mathbf{a}$ into the field $\Sigma f_i(\Lambda^{-1}(\mathbf{x} - \mathbf{a}))\, a_i$ (with a change of sign for improper transformations in the pseudoscalar case). Obviously the $\mathscr{C}$-valued fields of scalar (or pseudoscalar) type constitute a representation of the Poincaré group equivalent to the direct sum of 16 copies of its representation on scalar (or pseudoscalar) fields.

Given three real pseudoscalar fields $\pi_0 \equiv \pi_3, \pi_1$, and π_2, let us construct the $\mathscr{C}$-valued field of pseudoscalar type $\pi = \mathrm{i}(\pi_1\, e^{23} + \pi_2\, e^{31} + \pi_3\, e^{12})$. If ϕ is a Clifford field and γ^5 is defined as the constant Kahler field with value e^5 (so that it transforms into itself under proper Poincaré transformations and changes sign under improper transformations), the product

$$\gamma^5 \vee \phi \vee \pi \tag{22}$$

is a Clifford field (with the same definition as above of the action of a Poincaré transformation on a product of fields). Recalling the definition of the operators τ_α given in the previous section, the product (22) is identical with $\gamma^5 \vee \Sigma\, \pi_\alpha \tau_\alpha(\phi)$, and the function

$$L_{\rm int} = \mathrm{q}(\phi, \gamma^5 \vee \sum \pi_\alpha \tau_\alpha(\phi))$$

(where q is the coupling constant) is the familiar interaction Lagrangian of the nucleon field with the π-meson field (Bogoliubov and Shirkov, 1959, p. 411; Greiner and Muller, 1993, p. 91). From its equivalent expression

$$L_{\rm int} = \mathrm{q}(\phi, \gamma^5 \vee \phi \vee \pi))$$

one sees that the interaction Lagrangian is invariant under the transformation

(16) of the nucleon field ϕ, provided that at the same time the meson field π is subjected to the transformation

$$\pi \rightarrow u \vee \pi \vee u^{-1}$$

8. CONCLUSIONS

Our analysis and interpretation of the reduced Clifford equation has exhibited, for the nucleon, a relation between the external and internal symmetry groups whose essential features will now be summarized in terms which hint at a possible generalization.

We started from the following data:

• an associative algebra (now denoted generically by $\mathscr{A}$);

• a map of the homogeneous Lorentz group into the invertible elements of $\mathscr{A}$ defining, via left-multiplication, a linear representation of the group on $\mathscr{A}$ (multiple of an irreducible representation);

• a relativistic equation for the $\mathscr{A}$-valued fields admitting a Poincaré-invariant action.

Then the internal symmetry group presented itself as the group of invertible elements of $\mathscr{A}$ which leave the action invariant when acting on the fields by *right*-multiplication.

Finally, the algebraic structure of the representation space was exploited to supplement the field equation with coupling terms compatible with the transformation laws of the interacting fields under Poincaré transformations. In the two cases of known physical relevance considered, the structure required to define the coupling terms provided a decomposition of the fields into irreducible components corresponding to particles with different charges.

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